SuperformulaU

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When looking at the original "Superformula", described at www.geniaal.be

$$\text{Radius} = \left[\left| \frac{1}{a} \cos\left(\frac{m\phi}{4} \right) \right|^{n_1} + \left| \frac{1}{b} \sin\left(\frac{m\phi}{4} \right) \right|^{n_2} \right]^{\text{-}1/n} \tag{Superformula}$$

you can see, that the factor m is the same for sin and cos. Wouldn't it be useful to replace the m by two different factors as in the following equation?

$$\text{Radius} = \left[\left| \frac{1}{a} \cos\left(\frac{y\phi}{4} \right) \right|^{n_1} + \left| \frac{1}{b} \sin\left(\frac{z\phi}{4} \right) \right|^{n_2} \right]^{\text{-}1/n} \tag{2}$$

Analysis of the new parameters y and z

In the following, the parameters a, b, n_1 and n_2 are 1.

We define three kinds of structures:

- A structure is closed in infinity, when it has discontinuity and pole points but the limit from the left and right in those points is the same.

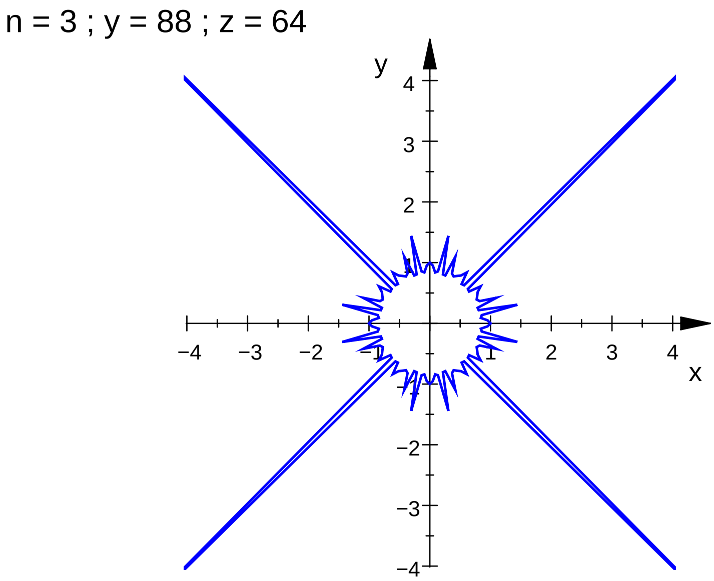

Figure 1: Structure, closed in infinity

- A structure is closed, if it has no poles and the limit from the left and right in the discontinuity points is the same.

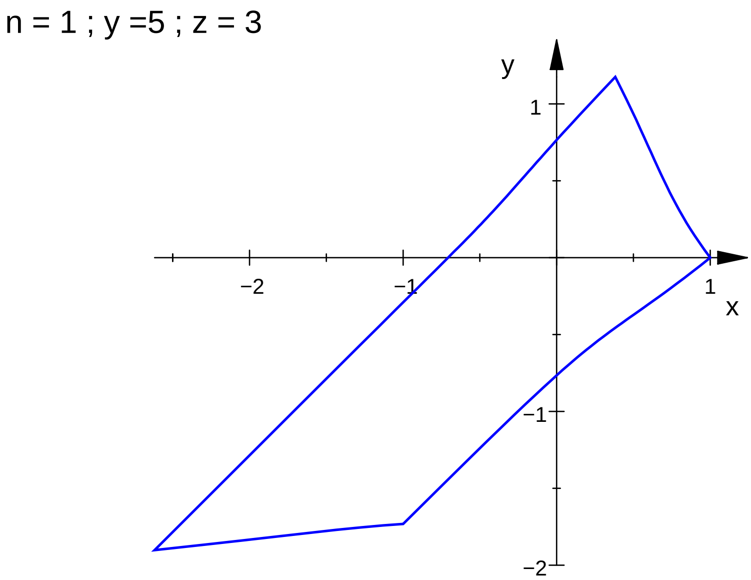

Figure 2: Closed structure; $\phi \in [0\,,\, 2\,\pi]$

- A structure is complete, if you can increase the polar angle without changing the structure.

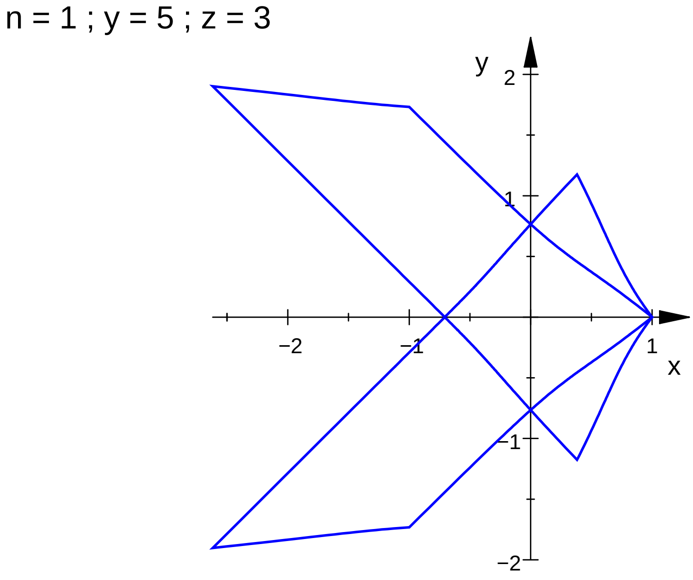

Figure 3: Complete structure ; $\phi \in [0\,,\,4\,\pi]$

When is a structure closed in infinity?

A structure is at least closed in infinity and complete, if y and z are rational. So we can define

$$y = \frac{a_y}{b_y} \qquad z = \frac{a_z}{b_z}$$

Equation (2) has poles, if

$$\frac{y\phi}{4} = \frac{(2m+1)\pi}{2} \quad and \quad \frac{z\phi}{4} = n\pi$$

where $m, n \in \mathbb{Z}$.
With this we get the constraint for poles:

$$\frac{2\,y}{z} = \frac{2\,m+1}{n} \tag{3}$$

and receive the following result: If a structure with rational y and z doesn't fulfil the constraint (3), it is closed. Otherwise it is closed in infinity.

What is the minimal angle to get a closed structure?

To answer this question, we have to solve the following equation

$$\cos\left(\frac{y\phi}{4}\right) + \sin\left(\frac{z\phi}{4}\right) = 1 \tag{4}$$

because we want to calculate the range of the angle ϕ. We start at $\phi = 0$, in this case the radius is 1. At the next angle, where (4) is fulfilled, the structure is closed. Unfortunately (4) could not be solved directly, but the solution can be found empiric, it is given in the answer of the next question.

Structures, which are closed in infinity are separated in several closed structures if n is switched to negative values (the structure is inverted). Because the infinite values turn to zero, see the following figure.

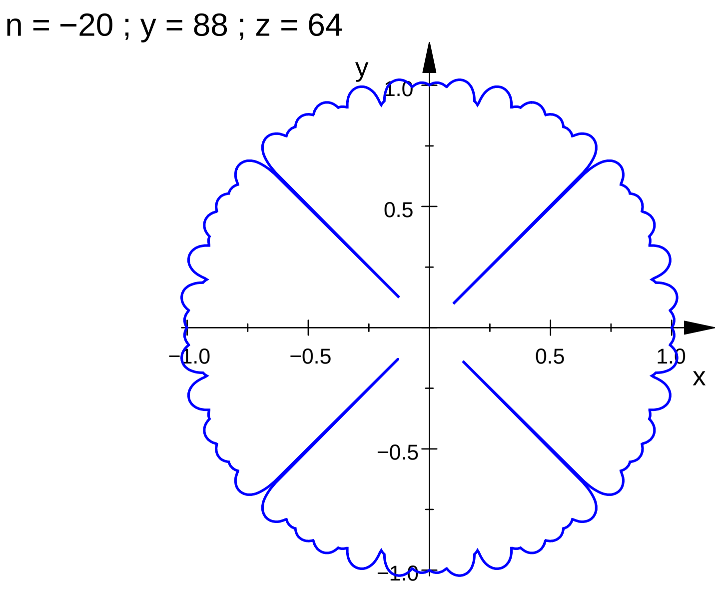

Figure 4: Inverted figure 1: four separate structures

What is the minimal angle to get a complete structure?

A structure is complete, if ϕ is at least $\phi_{\min}$.

$$\phi_{\min} = \begin{cases} b_y \cdot b_z \cdot 2\,\pi & \text{if} \quad \begin{array}{l} b_y,\ b_z \text{ are even} \quad \text{and} \quad a_y,\ a_z \text{ are odd} \\ b_y,\ b_z \text{ are odd} \quad \text{and} \quad a_y,\ a_z \text{ are even} \end{array} \\ b_y \cdot b_z \cdot 4\,\pi & \text{else} \end{cases} \tag{5}$$

(5) is a solution of (4) with the additional constraint, that the rise at $\phi = \epsilon$ and $\phi = \phi_{\min} - \epsilon$ is equal, where $\epsilon \in \mathbb{R} \rightarrow 0$. If a_y, a_z, b_y and b_z are odd, (4) is fulfilled but not the additional constraint. Therefore the structure is closed, but not complete in this case.

All complete structures have at least one symmetry-axis.

Examples of structures with the new parameters y and z

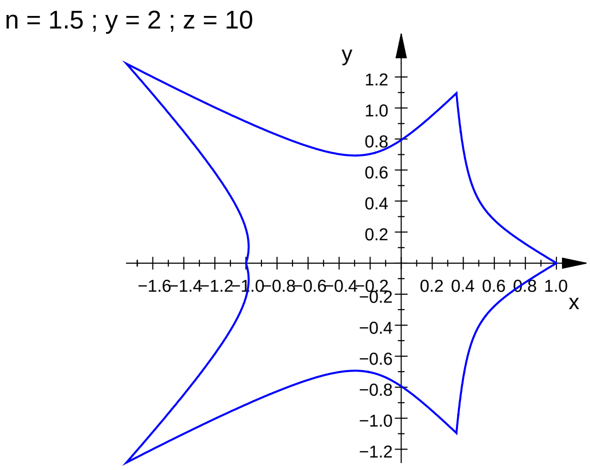

Figure 5: Distorted hexagon

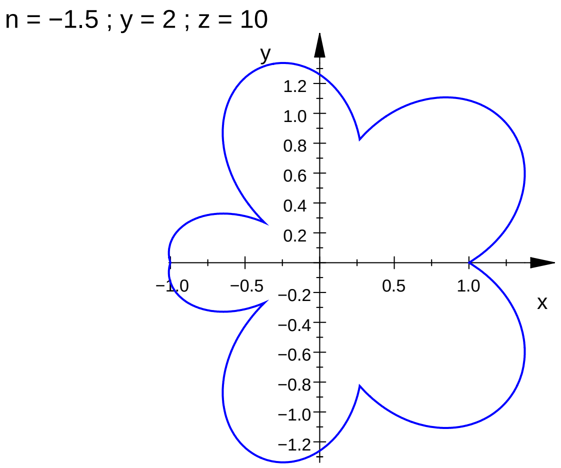

Figure 6: Inverted figure 5

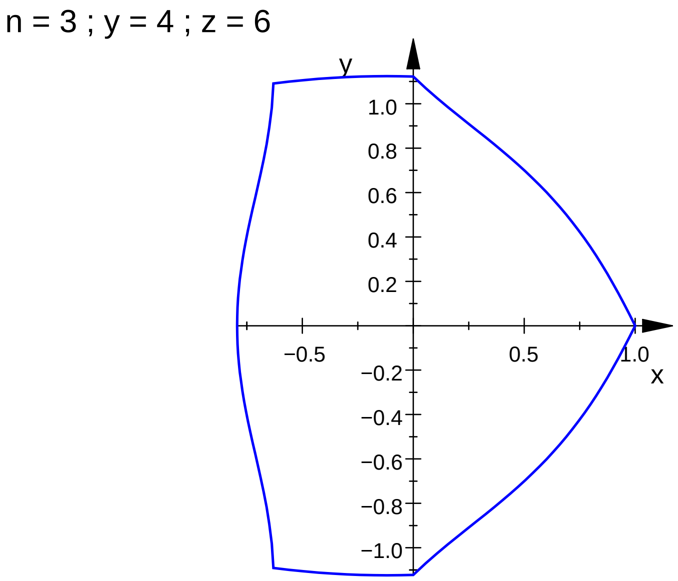

Figure 7: Pentagon

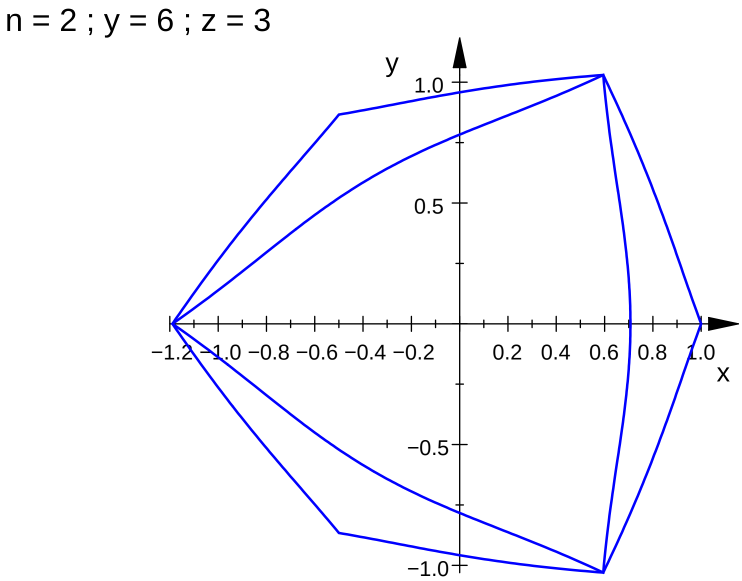

Figure 8: Nested structure; $\phi \in [0\,,\, 4\,\pi]$

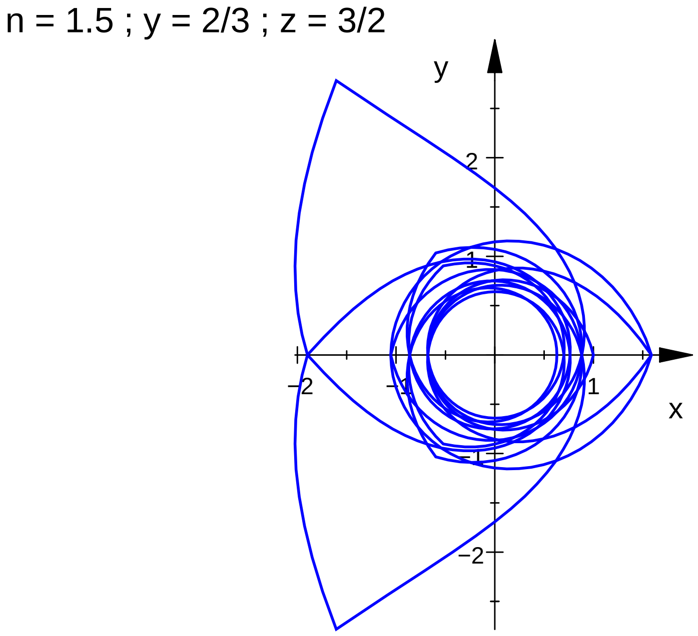

Figure 9: Another nested structure ; $\phi \in [0\,,\,24\,\pi]$

It is also possible to produce structures to archetypes of nature:

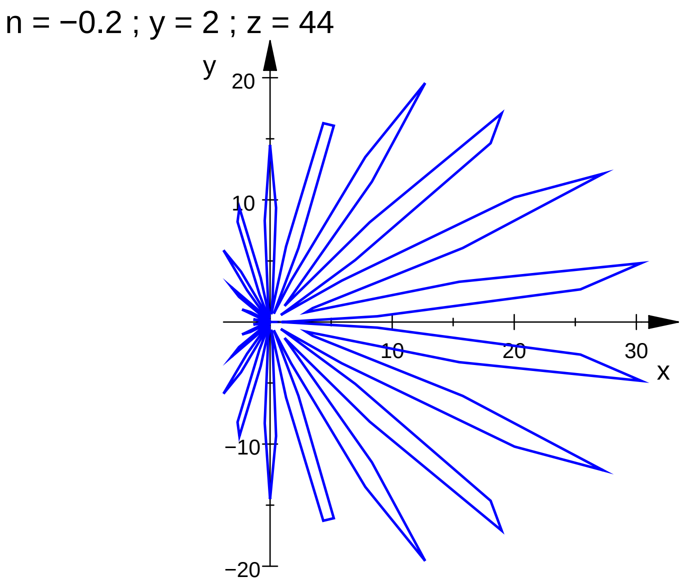

Figure 10: Leaf-like structure

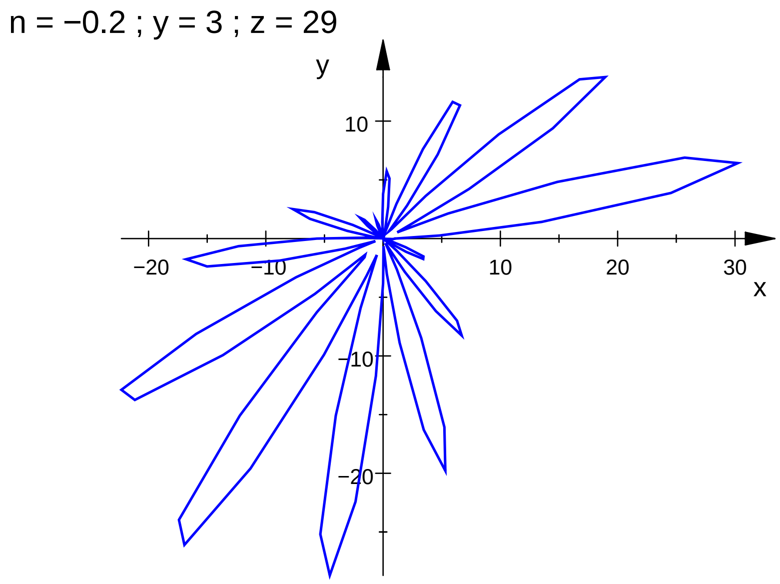

Figure 11: Another leaf-like structure

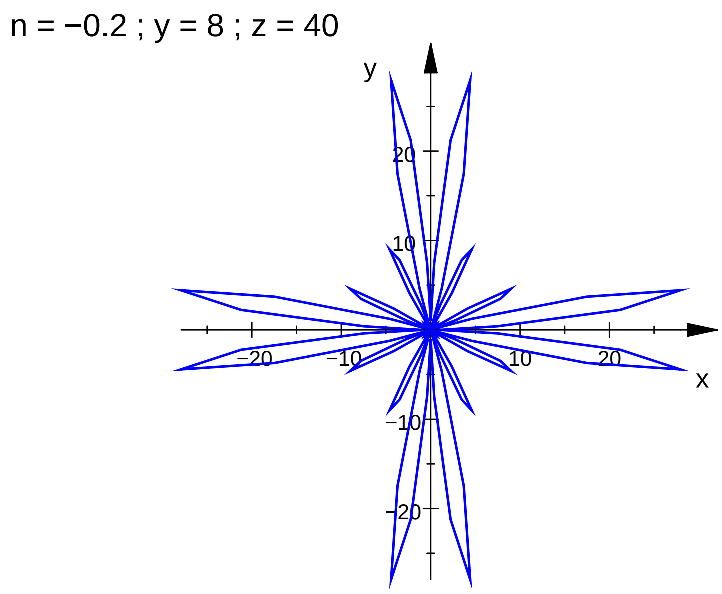

Figure 12: Frost-pattern-like structure

Equation (2) in three dimensions

To use the equation in three dimensions, we define:

$$R1 = \left[\left| \frac{1}{a} \cos\left(\frac{y\theta}{4}\right) \right|^{n_1} + \left| \frac{1}{b} \sin\left(\frac{z\theta}{4}\right) \right|^{n_2} \right]^{-1/n}$$

$$R2 = \left[\left| \frac{1}{a} \cos\left(\frac{y\phi}{4}\right) \right|^{n_1} + \left| \frac{1}{b} \sin\left(\frac{z\phi}{4}\right) \right|^{n_2} \right]^{-1/n}$$

and use spherical coordinates:

$$\begin{aligned} x &= R1 \cdot R2 \cdot \cos(\theta) \cos(\phi) \\ y &= R1 \cdot R2 \cdot \sin(\theta) \cos(\phi) \\ z &= R2 \cdot \sin(\phi) \end{aligned}$$

Whereas $\phi \in [0\,,\, 2m\,\pi]$; $\theta \in [0\,,\, m\,\pi]$ with $m \in \mathbb{N}$

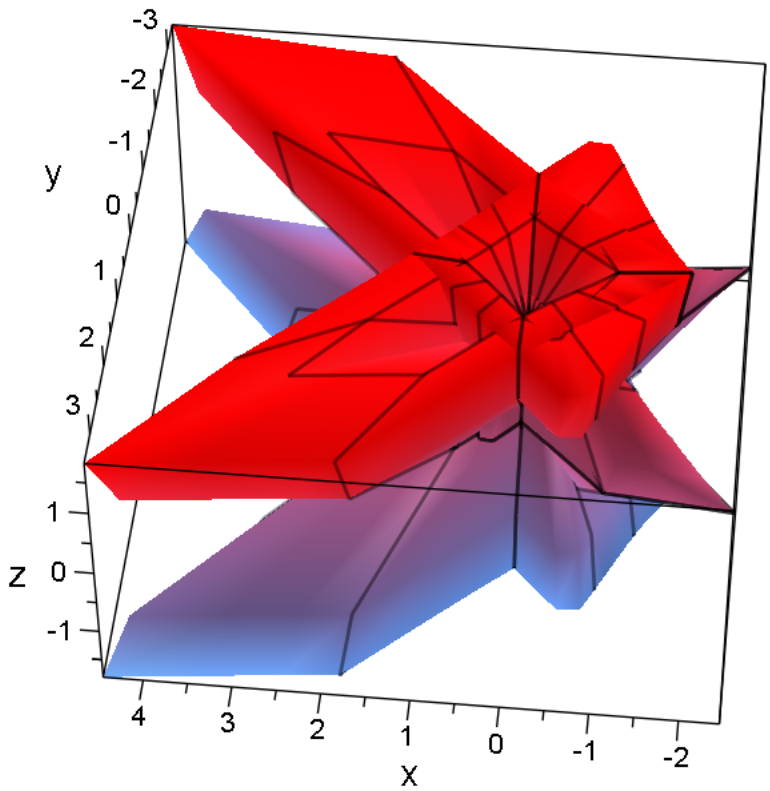

Figure 13: Figure 3 in three dimensions ; $\phi \in [0\,,\, 8\,\pi]$

The formulas (3) and (4) are also valid in three dimensions. But formula (5) changes to

$$\phi_{\min} = \begin{cases} b_y \cdot b_z \cdot 4\,\pi & \text{if} \quad \begin{array}{l} b_y,\ b_z \text{ are even} \quad \text{and} \quad a_y,\ a_z \text{ are odd} \\ b_y,\ b_z \text{ are odd} \quad \text{and} \quad a_y,\ a_z \text{ are even} \end{array} \\ b_y \cdot b_z \cdot 8\,\pi & \text{else} \end{cases} \tag{6}$$

All complete structures in three dimensions have at least two symmetry-planes.