The beauty behind x^{-x}

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1 Introduction

First in school, later at the university, I was confronted with the function

$$g(x) = x^x \tag{1}$$

The task the teacher/professor set was to differentiate it. The reason was that when one is able to do this, one really understood how to differentiate. It is also a nice example that 2 different differentiation approaches will lead to a result.

However, the function should be a simple exponential growth, but unintuitively it has a minimum. When looking closer one encounters more interesting and hard to explain facts. For example the spiral behavior for negative x.

Whenever I stumbled over $g(x)$ it in the past, I investigated a bit but I was always quickly stuck. As a PhD engineer I use math mainly as a tool and the function does not describe any problems I had to solve as engineer yet. Thus I never investigated deeper.

Recently, I wanted to use $g(x)$ to show my students how Euler's number is in everywhere and to use the function the way my teachers did. Since I could not answer questions an interested student might have when looking at the function, I investigated a bit since as teacher I should know more about the functions I use. This small paper is the result.

2 The reciprocal

The reciprocal of x^x turned out to be more interesting to unveil the mystery behind x^x. So we will mainly use it in the following.

$$f(x) = \frac{1}{x^x} = x^{-x} \tag{2}$$

Fig. 1 shows $g(x)$ and $f(x)$ in the interval $[0, 4]$.

$f(x)$ gives a hint for the question what rule with zero "has more power":

- $A^0 = 1$
- $0^B = 0$

Fig. 1 shows that for $\lim_{x\to 0+} f(x) = 1$. For $\lim_{x\to 0-} f(x)$ see Sec. 5.

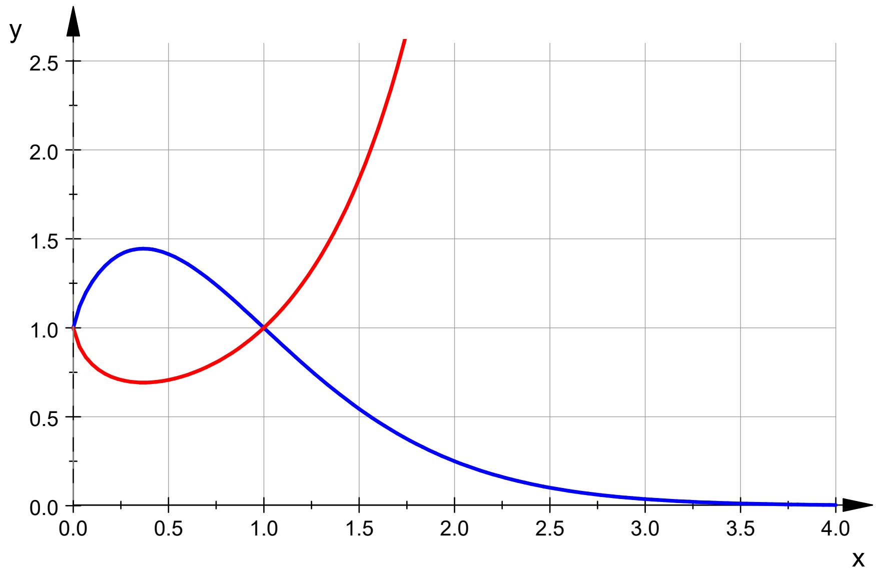


Figure 1: The functions $f(x)$ (blue) and $g(x)$ (red).

3 Euler's number and Derivative

Since $f(x)$ is differentiable, the minimum is quickly found:

$$x_{\min} = 1/e$$
$$y_{\min} = e^{1/e}$$

But why is there Euler's number? The reason is that one can express every exponent function with an exponential function:

$$x = \exp(\ln(x)) \Rightarrow x^{-x} = \exp(\ln(x)^{-x}) = \exp(-x\ln(x)) \tag{3}$$

So we get an exponential decay $e^{-a(x)}$ where a is

$$a(x) = x\ln(x) \tag{4}$$

$a(x)$ must have an extreme since for $x = 0$ and $x = 1$ $x \cdot \ln(x) = 0$, see also Fig. 2.

Calculating the derivative is in so far interesting as when moving towards zero, it does not converge to a maximal slope but the slope goes to infinity, see Fig. 3.

Looking at the derivative:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}f(x) =& \frac{\mathrm{d}}{\mathrm{d}x}e^{\ln(x)^{-x}} = \frac{\mathrm{d}}{\mathrm{d}x}e^{-x\ln(x)}\\ =& e^{-x\ln(x)} \cdot (-\ln(x) - 1)\\ =& -x^{-x}\,(1 + \ln(x))\end{aligned} \tag{5}$$

this is clear. But it is not intuitive when just looking at the graph of $f(x)$.

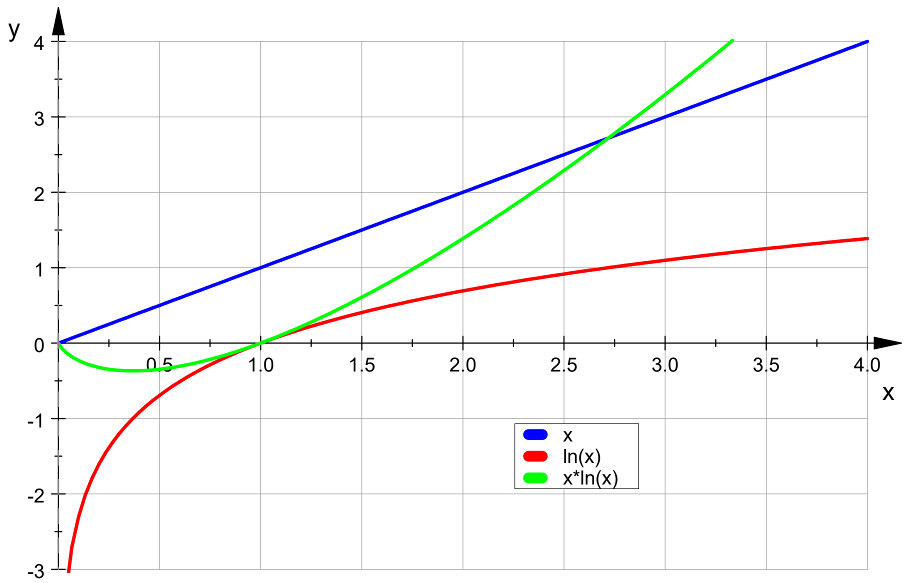


Figure 2: The functions x, $\ln(x)$ and $x\ln(x)$.

4 Integral

The next interesting fact is that the integral from zero to infinity is almost 2. Once I did this 20 years back numerically, I thought the result is just a numerical rounding issue. But even with high precision and with different CAS systems, the result is:

$$\int_0^\infty x^{-x}\,\mathrm{d}x \approx 1.9954559575 \tag{6}$$

so definitively lower than 2.

For the fun with my name and its German meaning I gave it the name "Gestöhrte" constant.

4.1 Attempt to Integrate

To integrate, the x can be written as $x = e^{\ln(x)}$. By using now the definition (Taylor series) of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{7}$$

the integral can be rewritten as:

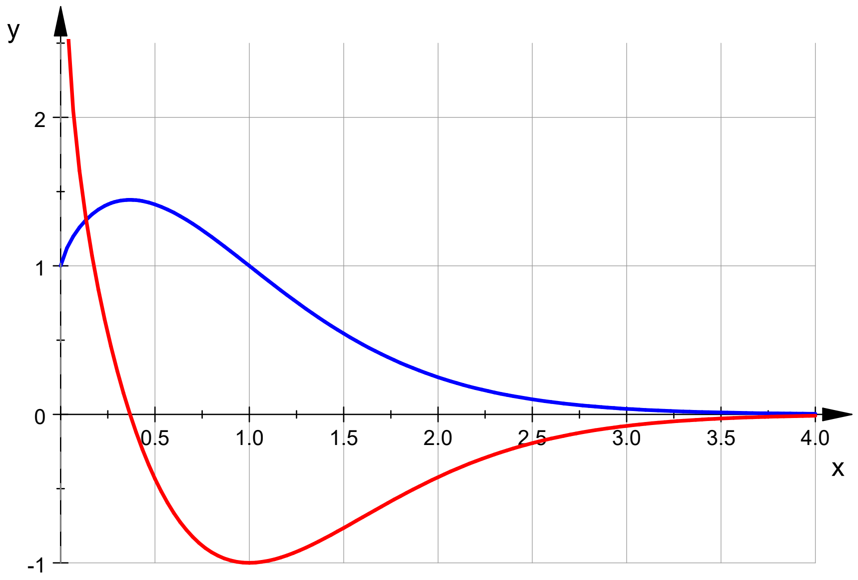


Figure 3: The functions $f(x)$ (blue) and $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$ (red).

$$\begin{aligned}\int_0^\infty x^{-x}\,\mathrm{d}x &= \int_0^\infty \mathrm{e}^{-x\ln(x)}\,\mathrm{d}x\\ &= \int\limits_0^\infty \sum_{n=0}^\infty \frac{(-x\ln(x))^n}{n!}\,\mathrm{d}x\\ &= \sum_{n=0}^\infty \frac{-1^n}{n!}\int\limits_0^\infty x^n \ln(x)^n\,\mathrm{d}x \end{aligned} \tag{8}$$

To solve now

$$\int_0^\infty x^n \ln(x)^n\,\mathrm{d}x \tag{9}$$

one approach is to change the integrand to $u = \ln(x)$, thus $x = \mathrm{e}^u$ and $\mathrm{d}u = \frac{1}{x} = \mathrm{e}^{-u}$:

$$\begin{aligned}&\int_{-\infty}^\infty \mathrm{e}^{un}u^n\mathrm{e}^{-u}\,\mathrm{d}u\\ &\int_{-\infty}^\infty \mathrm{e}^{u(n-1)}u^n\,\mathrm{d}u\end{aligned} \tag{10}$$

For the special case that one would like to know

$$\int_0^1 x^{-x}\,\mathrm{d}x \tag{11}$$

one can use the trick Mr. Bazett uses in this video – to transform (10) into the Gamma function. The result is

$$\int_0^1 x^{-x}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{1}{(n-1)^{n-1}} \approx 1.2913 \tag{12}$$

The Feynman method of defining a superset of integral functions in the form

$$I(t) = \sum_{n=0}^{\infty} \frac{-1^n}{n!} \int\limits_0^{\infty} \mathrm{e}^{ut(n-1)} u^n \,\mathrm{d}u \tag{13}$$

does not help much.

Using the CAS program MuPad to rewrite (10) also gives no beneficial result:

$$\int_{-\infty}^{\infty} \mathrm{e}^{u(n-1)} u^n \,\mathrm{d}u = \frac{1}{-(1-n)^{n+1}} \int_{u(1-n)}^{\infty} \mathrm{e}^{-t} t^n \,\mathrm{d}t \tag{14}$$

Another way is to solve (9) by partial integration n-times and then looking if there is a repeating pattern for each step:

$$\int_a^b \underbrace{x^n}_{\hat{=}\mathrm{d}r} \underbrace{\ln(x)^n}_{\hat{=}s} \,\mathrm{d}x = r \cdot s - \int_a^b r \,\mathrm{d}s \tag{15}$$

$$\begin{aligned} r &= \frac{x^{n+1}}{n+1} \\ s &= \ln(x)^n \\ \mathrm{d}s &= \frac{n \ln(x)^{n-1}}{x} \mathrm{d}x \end{aligned}$$

This delivers the first integration step $k = 0$:

$$\begin{aligned} \left\{ r \cdot s - \int_a^b r \,\mathrm{d}s \right\}_{k=0} &= \left| \frac{x^{n+1}}{n+1} \cdot \ln(x)^n \right|_a^b - \int_a^b \frac{x^{n+1}}{n+1} \cdot \frac{n \ln(x)^{n-1}}{x} \mathrm{d}x \\ &= \left| \frac{x^{n+1}}{n+1} \cdot \ln(x)^n \right|_a^b - \int_a^b \frac{x^n}{n+1} n \ln(x)^{n-1} \,\mathrm{d}x \end{aligned} \tag{16}$$

Step $k = 1$ is:

$$\int_a^b \underbrace{\frac{x^n}{n+1}}_{\hat{=}\mathrm{d}r} \underbrace{n\ln(x)^{n-1}}_{\hat{=}s} \,\mathrm{d}x = \left\{ r \cdot s - \int_a^b r \,\mathrm{d}s \right\}_{k=1} \tag{17}$$

$$\begin{aligned} r &= \frac{x^{n+1}}{(n+1)^2} \\ s &= n\ln(x)^{n-1} \\ \mathrm{d}s &= \frac{n(n-1)\ln(x)^{n-2}}{x}\mathrm{d}x \Rightarrow x \text{ will always cancel out with } r \end{aligned}$$

$$\left\{ r \cdot s - \int_a^b r \,\mathrm{d}s \right\}_1 = \left| \frac{x^{n+1}}{(n+1)^2} \cdot n\ln(x)^{n-1} \right|_a^b - \int_a^b \frac{x^n}{(n+1)^2} \cdot n(n-1)\ln(x)^{n-2}\,\mathrm{d}x \tag{18}$$

So the first 2 steps give this equation:

$$\int_a^b x^n \ln(x)^n \,\mathrm{d}x = \left| \frac{x^{n+1}}{n+1} \cdot \ln(x)^n \right|_a^b - \left| \frac{x^{n+1}}{(n+1)^2} \cdot n\ln(x)^{n-1} \right|_a^b \tag{19}$$

$$+ \int_a^b \frac{x^n}{(n+1)^2} \cdot n(n-1)\ln(x)^{n-2}\,\mathrm{d}x \tag{20}$$

We can now derive the repeating pattern for every integration step. The tricky part is hereby to see that

$$1, n, n(n-1), n(n-1)(n-2) = \frac{n!}{(n-k)!} \tag{21}$$

In effect, partial integration $k = n$-times results in this sum

$$\int_a^b x^n \ln(x)^n \,\mathrm{d}x = \sum_{k=0}^{n} \left| (-1)^k \frac{x^{n+1}\ln(x)^{n-k} n!}{(n+1)^{k+1} \cdot (n-k)!)} \right|_a^b \tag{22}$$

Going back to our initial integral (8) we can write:

$$\begin{aligned} \int_0^\infty x^{-x}\,\mathrm{d}x &= \sum_{n=0}^{\infty} \frac{-1^n}{n!} \int\limits_0^\infty x^n \ln(x)^n \,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{-1^n}{n!} \sum_{k=0}^{n} \left| (-1)^k \frac{x^{n+1}\ln(x)^{n-k} n!}{(n+1)^{k+1} \cdot (n-k)!)} \right|_0^\infty \\ &= \sum_{n=0}^{\infty} -1^n \sum_{k=0}^{n} \left| (-1)^k \frac{x^{n+1}\ln(x)^{n-k}}{(n+1)^{k+1} \cdot (n-k)!)} \right|_0^\infty \end{aligned} \tag{23}$$

When we set $a = 0$ in (22), we see that the term with $x = a$ becomes zero. Thus we can write

$$\int_0^b x^{-x}\,\mathrm{d}x = \sum_{n=0}^{\infty} -1^n \sum_{k=0}^{n} (-1)^k \frac{b^{n+1}\ln(b)^{n-k}}{(n+1)^{k+1}\cdot(n-k)!)} \tag{24}$$

Looking at Fig. 1 we see that we get a good approximation for the desired integral by using $b = 5$. Using a CAS, we get:

$$\begin{aligned}\int_0^5 x^{-x}\,\mathrm{d}x &= \sum_{n=0}^{\infty} -1^n \sum_{k=0}^{n} (-1)^k \frac{5^{n+1}\ln(5)^{n-k}}{(n+1)^{k+1}\cdot(n-k)!)} \\ &\approx \sum_{n=0}^{30} -1^n \sum_{k=0}^{n} (-1)^k \frac{5^{n+1}\ln(5)^{n-k}}{(n+1)^{k+1}\cdot(n-k)!)} \\ &\approx 1.99533\end{aligned}$$

It is hereby interesting that the sum first results in a permanently positive value when summing at least $n = 14$ terms. For larger b the greater the number of n terms has to be, see Fig. 4.

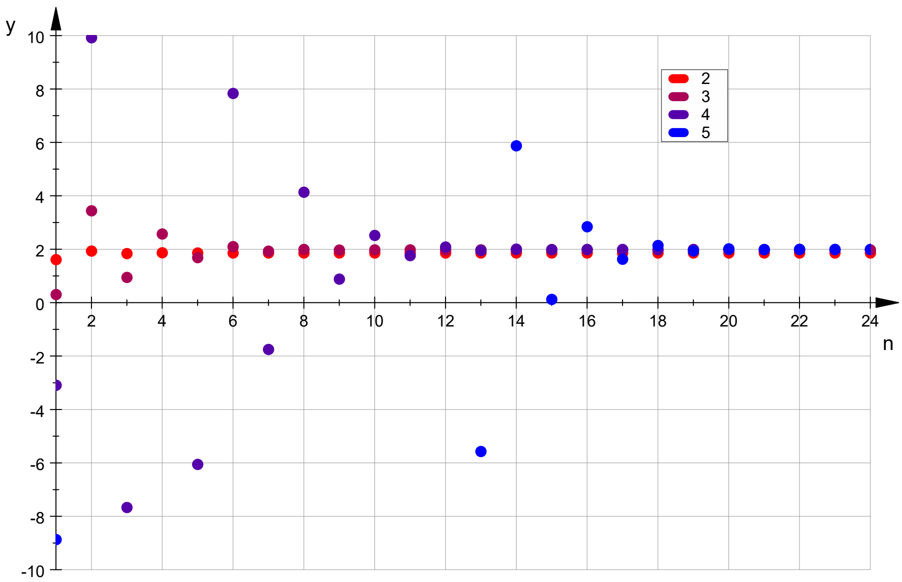


Figure 4: (24) for $n = [1; 24]$ and $b = [2, 3, 4, 5]$.

Just to cross-check with (12), we get indeed (with $\lim_{b\to 1} b$) :

$$\begin{aligned}\int_0^1 x^{-x}\,\mathrm{d}x &\approx \sum_{n=0}^{5} -1^n \sum_{k=0}^{n} (-1)^k \frac{1.0001^{n+1}1\ln(1.0001)^{n-k}}{(n+1)^{k+1}\cdot(n-k)!)} \\ &\approx 1.2913\end{aligned}$$

However, in effect the symbolic integration has no benefit than to compute the integral numerically. We did not unveil any insights about the nature of the function x^{-x}.

5 x^{-x} in complex space

It is obvious that x^{-x} is interesting with complex numbers as only for $x > 0$ the imaginary part is always zero. So in a x-$f(x)$-diagram one gets only real values at $x = -1, -2, \ldots$. But what happens in between?

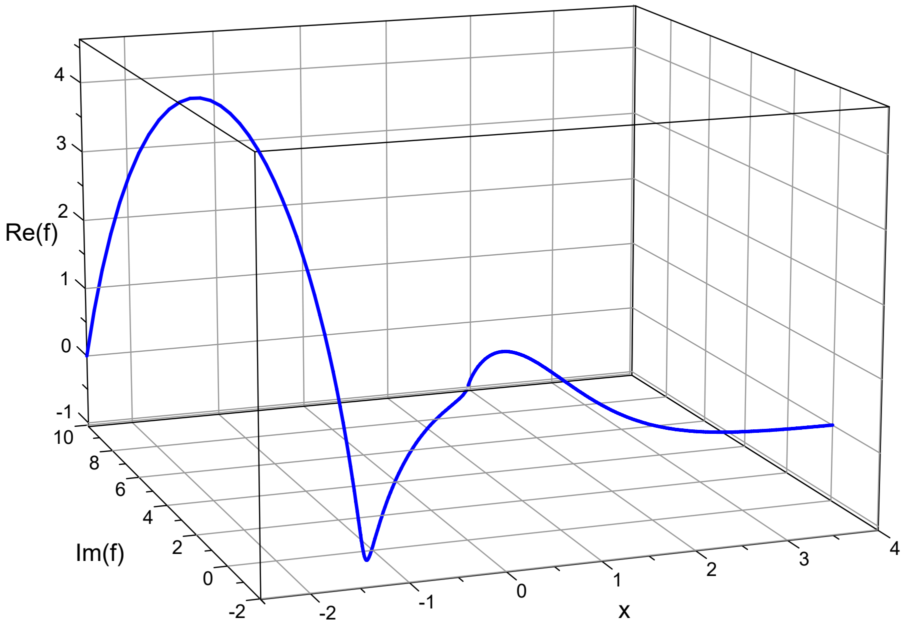


Figure 5: The function $f(x)$ in the complex space. The y-axis is $\Im(f(x))$, the z-axis is $\Re(fx)$.

Fig. 5 shows x^{-x} in the complex space. We can now also answer the question from Sec. 2 that the rule $A^0 = 1$ "has more power" than $0^B = 0$ since $\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = 1$. This is not surprising since

$$A^B = \exp(B \cdot \ln(A)) \quad (25)$$
$$0^0 = \exp(0 \cdot \underbrace{\ln(0)}_{\to -\infty})$$

, so the term in the exponential is zero, and thus we get a 1.

One can see in Fig. 5 that along the x-axis the function forms a spiral.

A spiral follows these equations:

$$x_{\text{spiral}}(\phi) = r(\phi)\cos(\phi) \quad (26)$$
$$y_{\text{spiral}}(\phi) = r(\phi)\sin(\phi) \quad (27)$$

Trying to determine what $r(\phi)$ is, one notices that every $x = 0.5$ there is a turn by $\phi = \pi/2$, see Fig. 6.

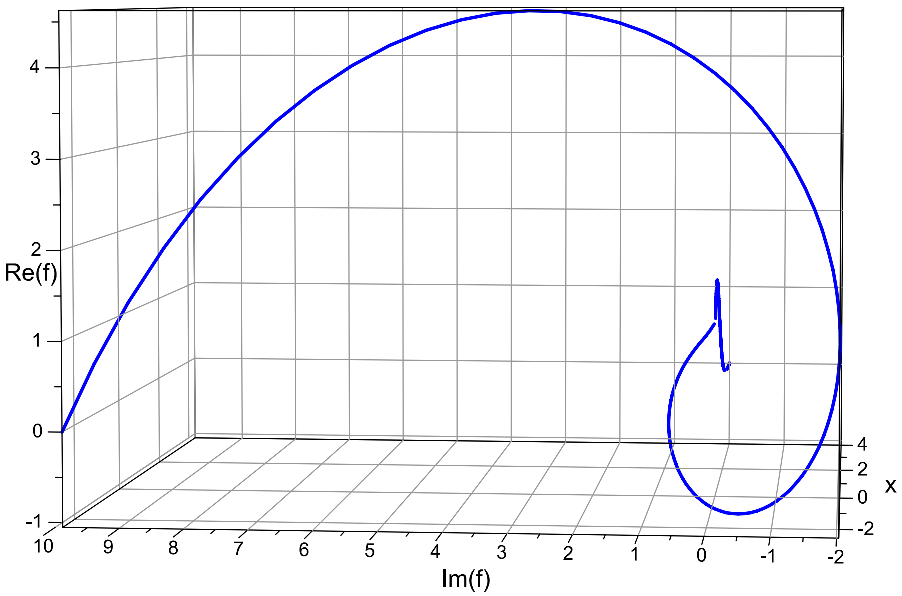


Figure 6: The function $f(x)$ in the complex space, viewed in the x-direction.

At $\phi = 0$: $x = 0$ and thus $f(0) = 1 + 0\,\mathrm{i}$.

At $\phi = \pi/2$: $x = -0.5$ and thus $f(-0.5) = 0 + \sqrt{0.5}\,\mathrm{i}$.

At $\phi = \pi$: $x = -1$ and thus $f(-1) = -1 + 0\,\mathrm{i}$.

At $\phi = 1.5\pi$: $x = -1.5$ and thus $f(-1.5) = 0 - \sqrt{3.375}\,\mathrm{i}$.

This comes unexpected. Why 3.375? Why is this a number in such a "fundamental" function like x^{-x}?

Moving on:

At $\phi = 2\pi$: $x = -2$ and thus $f(-2) = 4 + 0\,\mathrm{i}$.

At $\phi = 2.5\pi$: $x = -2.5$ and thus $f(-2.5) = 0 + \sqrt{97.65625}\,\mathrm{i}$.

At $\phi = 3\pi$: $x = -3$ and thus $f(-3) = 27 + 0\,\mathrm{i}$.

At $\phi = 3.5\pi$: $x = -3.5$ and thus $f(-3.5) = 0 - \sqrt{6433.9296875}\,\mathrm{i}$.

So what is happening here?

5.1 Logarithm of negative numbers

Since $x^{-x} = \exp(-x\ln(x))$, when we e.g. input $x = -1.5$, this triggers the computation of $\ln(-1.5)$. But what is the logarithm of a negative number?

$\ln(x)$ delivers the solution to the equation

$$e^y = x \tag{28}$$

$$y = \ln(x) \tag{29}$$

So we can ask what y we get for

$$e^y = -1.5 \tag{30}$$

This is a sensible question and there are solutions for y.[1]

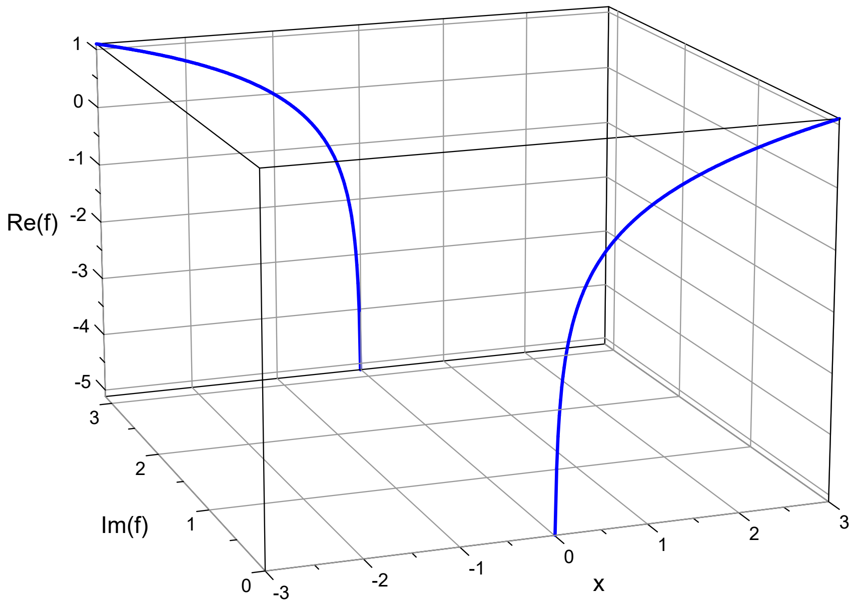


Figure 7: The function $\ln(x)$ in the complex space.

To get an idea quickly we can simply plot $\ln(x)$ in the complex space. This is Fig. 7. The surprising result is that for $x < 0$ the logarithm adds a constant offset in the imaginary axis by πi. So we can write:

$$\ln(x) = \begin{cases} \ln(-x) + \pi\mathrm{i} & \text{for } x < 0 \\ \ln(x) & \text{for } x > 0 \end{cases} \tag{31}$$

When we solve now (30) with this formula:

$$y = \ln(-1.5) = \ln(1.5) + \pi\mathrm{i} \tag{32}$$

$$e^y = e^{\ln(1.5)+\pi\mathrm{i}} \tag{33}$$

$$= e^{\ln(1.5)} \cdot \underbrace{e^{\pi\mathrm{i}}}_{\text{Euler's identity}} \tag{34}$$

$$= 1.5 \cdot -1 \tag{35}$$

So (31) solves indeed (30).[2]

[1]For details about the complex logarithm, see the corresponding Wikipedia article: Complex logarithm.

[2]Actually there are infinity many solutions since $e^{\pi\mathrm{i}}$ is actually a rotation by 180° around the $\Re(x)$-axis in the complex space. Therefore all $e^{(2k+1)\pi\mathrm{i}} = -1$, $k = 1, 2, 3\ldots$ are solutions.

5.2 Explanation of the spiral

The formula (31) explains why $f(x)$ spirals for $x < 0$:
Remember that $x^{-x} = \exp(-x\ln(x))$. For small x, $\exp(-x)$ growths. So the real part of $f(x)$ growths for smaller x. The logarithm inside the exponential adds to the growth a "push" for the the imaginary part. This means for the function $-x\ln(x)$ and $x < 0$:

$$-x\ln(x) = -x\ln(-x) - x\pi\mathrm{i} \tag{36}$$

$-x\ln(x)$ in the complex space is shown in Fig. 8. One can see that the function is pushed along the imaginary axis.

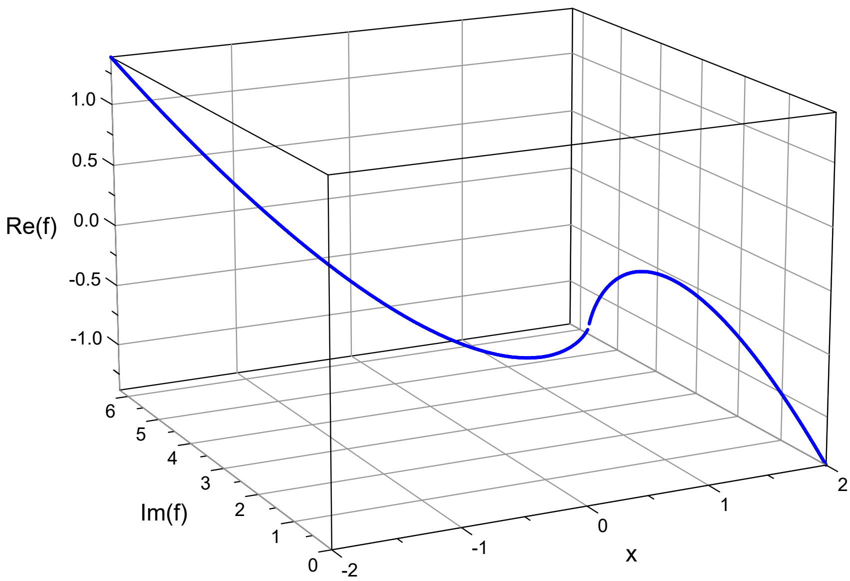


Figure 8: The function $-x\ln(x)$ in the complex space.

However, $f(x) = \exp(-x\ln(x))$ and this is way the there is a spiral:

$$\exp(-x\ln(-x) - x\pi\mathrm{i}) = \underbrace{\exp(-x\ln(-x))}_{\text{exponential growth}}\underbrace{\exp(-x\pi\mathrm{i})}_{\text{helix}} \tag{37}$$

The term $\exp(-x\pi\mathrm{i})$ brings in the rotation. To visualize this, we plot

$$h(x) = \exp(-x\pi\mathrm{i}) \tag{38}$$

in the complex space, see Fig. 9. So with every $x = 1$-step we get a half-turn around the x-axis in the plot.

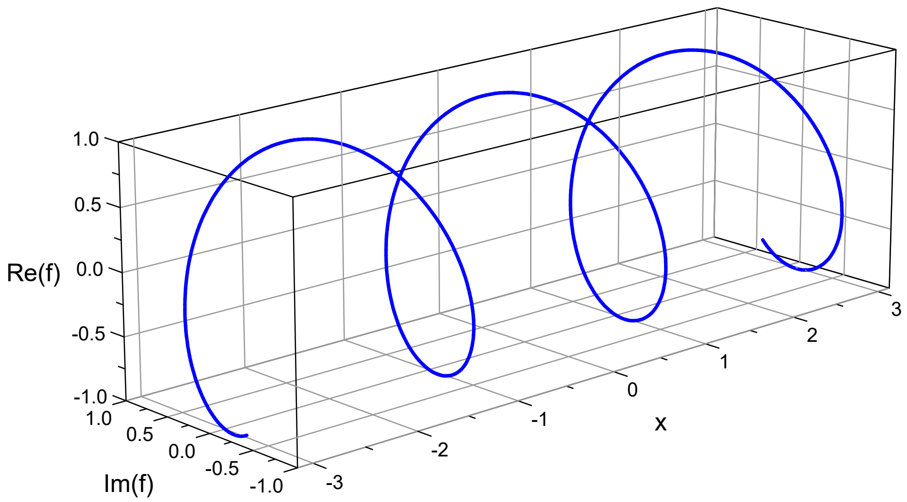


Figure 9: The function $h(x) = \exp(-x\pi\mathrm{i})$ in the complex space.

This corresponds directly to the identity

$$\exp(\mathrm{i}x) = \cos(x) + \mathrm{i}\sin(x) \tag{39}$$

$$\begin{aligned}\exp(-x\pi\mathrm{i}) &= \cos(-x\pi) + \mathrm{i}\sin(-x\pi)\\ &= \cos(x\pi) - \mathrm{i}\sin(x\pi)\end{aligned} \tag{40}$$

So the logarithm in $f(x)$ brings the shift into the imaginary axis and the exponential function around the logarithm transforms it to a permanent rotation around the x-axis.

We can now also compute the “mysterious” $f(-1.5) = -\sqrt{3.375}\,\mathrm{i}$:

$$\begin{aligned}f(x) &= \exp\left(-x\ln(x)\right)\\ f(x<0) &= \exp\left(-x\ln(-x) - x\pi\mathrm{i}\right)\\ &= \exp\left(-x\ln(-x)\right)\cdot\exp(-x\pi\mathrm{i})\\ f(-1.5) &= \exp\left(1.5\ln(1.5)\right)\cdot\underbrace{\exp(1.5\pi\mathrm{i})}_{=-\mathrm{i}}\\ &= -1.5^{1.5}\,\mathrm{i}\\ f(-1.5) &= -\sqrt{\underbrace{1.5^3}_{3.375}}\,\mathrm{i}\end{aligned} \tag{41}$$

Since we know that $(-1)^1 = -1$ and $(-2)^2 = 4$ we can cross-check our formula:

$$\begin{aligned}f(-1) &= \exp\left(1\ln(1)\right)\cdot\underbrace{\exp(-1\pi\mathrm{i})}_{=-1}\\ f(-1) &= -1^1 = -1\end{aligned}$$

$$f(-2) = \exp\left(2\ln(2)\right) \cdot \underbrace{\exp(-2\pi\mathrm{i})}_{=1}$$

$$f(-2) = 2^2 = 4$$

Finally, we can also specify the growth function of the spiral according to (27):

$$\Re\left(f\left(x = \frac{\phi}{\pi}\right)\right) = r(\phi)\cos(\phi)$$

$$\Im\left(f\left(x = \frac{\phi}{\pi}\right)\right) = r(\phi)\sin(\phi)$$

$$r(\phi) = \exp\left(\frac{\phi}{\pi}\ln\left(\frac{\phi}{\pi}\right)\right) \tag{42}$$

With $r(\phi)$ we can be creative. So for the family of functions

$$\exp\left(-ax\ln(x)\right) = x^{(-ax)} \tag{43}$$

we can set $0 < a < 1$ and get spirals with a lower growth rate.

We can also create spirals that converge to zero. Plotting (37) in the complex plane for $x \geq 0$ gives Fig. 10. This is a "spiral" whose radius increases for $0 < x < 1/\mathrm{e}$ and for $x > 1/\mathrm{e}$ its radius decreases to zero – exactly as $f(x)$ does. To get a real spiral whose radius only decreases with every turn, we can plot it for $x \geq 1$.

Derived from (37) one can take the family of functions $\exp(-ax\ln(-bx) - x\pi\mathrm{i})$ for $x > 0$ and play around.

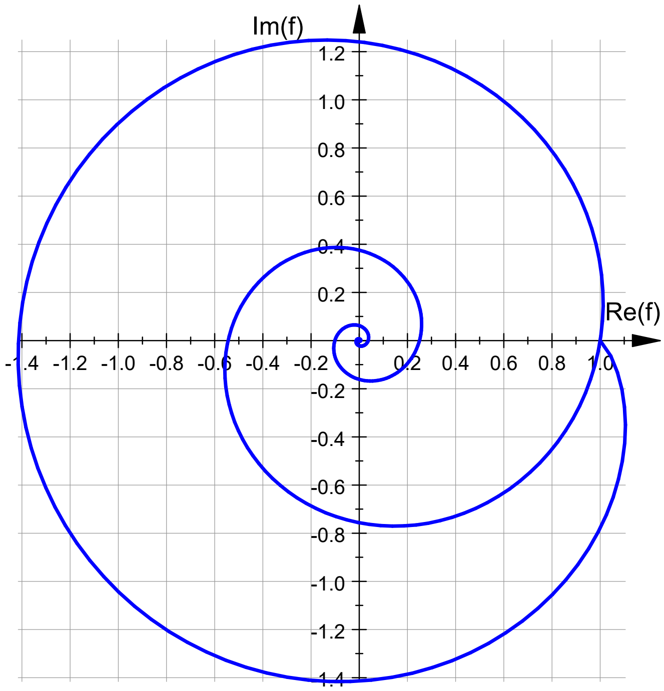


Figure 10: The function $\exp(-x\ln(-x) - x\pi\mathrm{i})$ for $x > 0$ in the complex plane.

6 It can be complex in the complex space

With the knowledge from the previous section we can now play a bit around to create functions that look nice in the complex space.

6.1 $x^{-\ln(x)}$ in complex space

$$
\begin{aligned}
p(x) = x^{-\ln(x)} &= \exp\left(-\left(\ln(x)\right)^2\right) \\
p(x<0) &= \exp\left(-\left(\ln(-x) + x\pi\mathrm{i}\right)^2\right) \\
&= \exp\left(-\left(\left(\ln(-x)\right)^2 + 2\ln(-x)x\pi\mathrm{i} - x^2\pi^2\right)^2\right) \qquad (44)
\end{aligned}
$$

So there is a rotation but it is hard to see from the formula what exactly will happen. But thanks to our knowledge of (31), we can plot $p(x)$ for $x > 0$, see Fig. 11, to predict the appearance for $x < 0$: There will be a spiral with the maximum at $x = -1$ that decays for smaller x. $x^{-\ln(x)}$ in complex space is shown in Fig. 12.

6.2 $\exp\left(-x^{-x}\right)$ in complex space

Using the same technique like for $x^{-\ln(x)}$ and plotting $\exp\left(-x^{-x}\right)$ for $x > 0$, see Fig. 11, we would predict: First a spiral then for smaller x a helix. No spiral since for small x

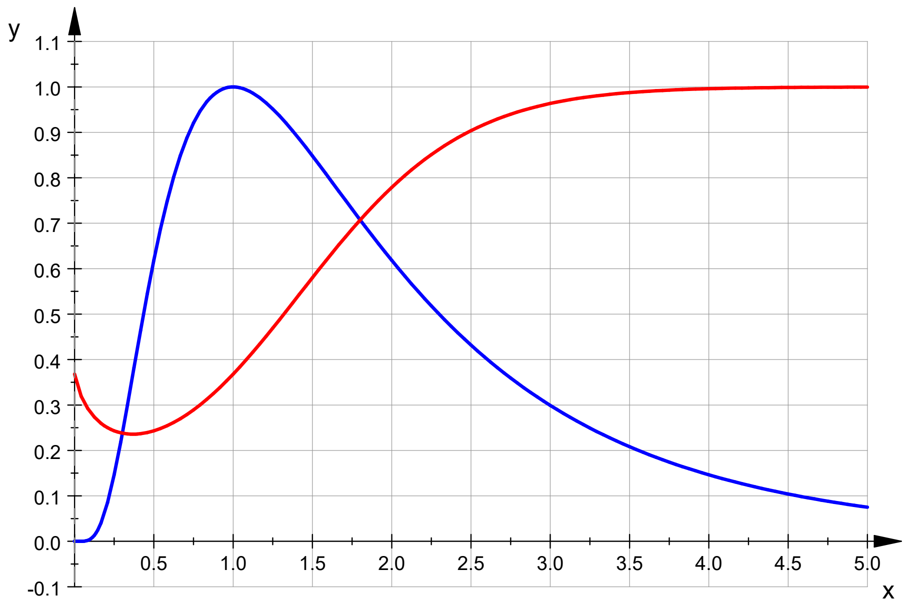


Figure 11: The functions $p(x) = x^{-\ln(x)}$ (blue) and $\exp\left(-x^{-x}\right)$ (red).

we get as result always a 1.

But it looks actually different, see Fig. 13. First the expected spiral, but then from $x = [2, 2.5]$ almost no spiral. At $x = 2.58$ it starts to spiral with its maximum around $x = \pi$. Then it decays to zero. So our prediction was not true.

Rewriting $\exp\left(-x^{-x}\right)$ the usual way leads us to:

$$\exp\left(-x^{-x}\right) = \exp\left(-\exp\left(-x\ln(x)\right)\right) \tag{45}$$

using now (36) leads us to

$$\exp\left(-\exp\left(-x\ln(x)\right)\right) = \exp\left(-\exp(-x\ln(-x) - x\pi\mathrm{i})\right) \tag{46}$$

$$= \exp\left(-\underbrace{\exp(-x\ln(-x))}_{\text{exponential growth}}\underbrace{\exp(-x\pi\mathrm{i})}_{\text{helix}}\right) \tag{47}$$

So for small x the exponential growth goes to ∞ and since $\exp(-\infty) = 0$ we should get a zero. This is what we see but for the range $x = [0, 4]$ it is hard to make a prediction just by looking at (47).

By the way, it is interesting to see how $\exp\left(\exp(-x\pi\mathrm{i})\right)$ gives a distorted helix, see Fig. 14.

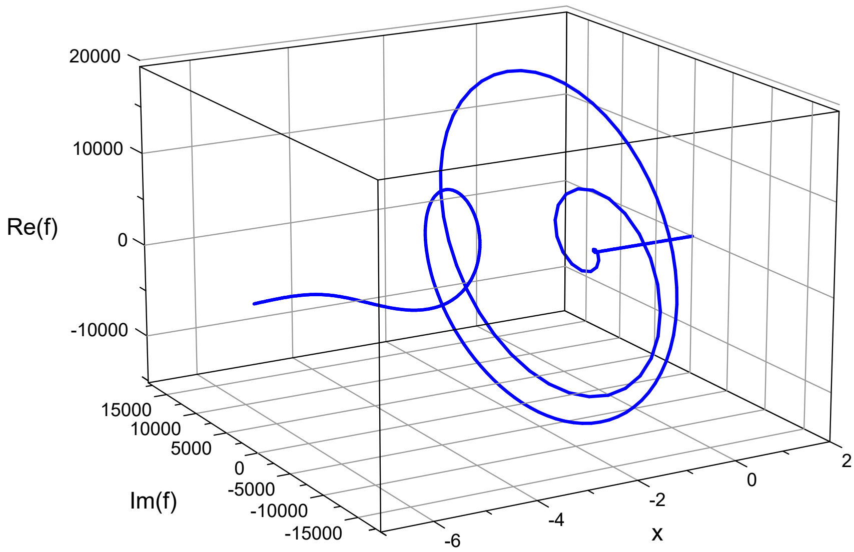
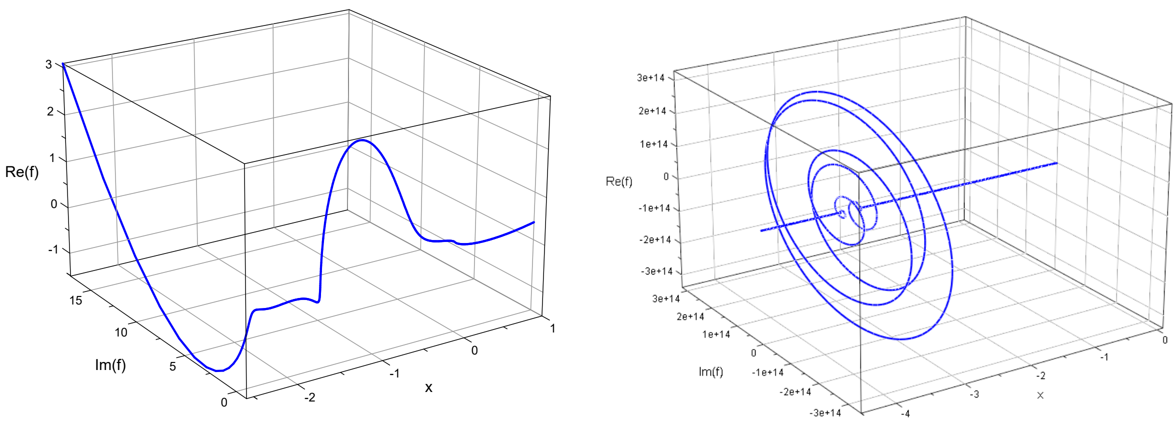


Figure 12: The function $p(x)$ in the complex space.



(a) The function $\exp\left(-x^{-x}\right)$ in the interval $x = [-2.58, 1]$.

(b) The function $\exp\left(-x^{-x}\right)$ in the interval $x = [-4, 0.1]$.

Figure 13: The function $\exp\left(-x^{-x}\right)$ and in the complex space for different intervals of x.

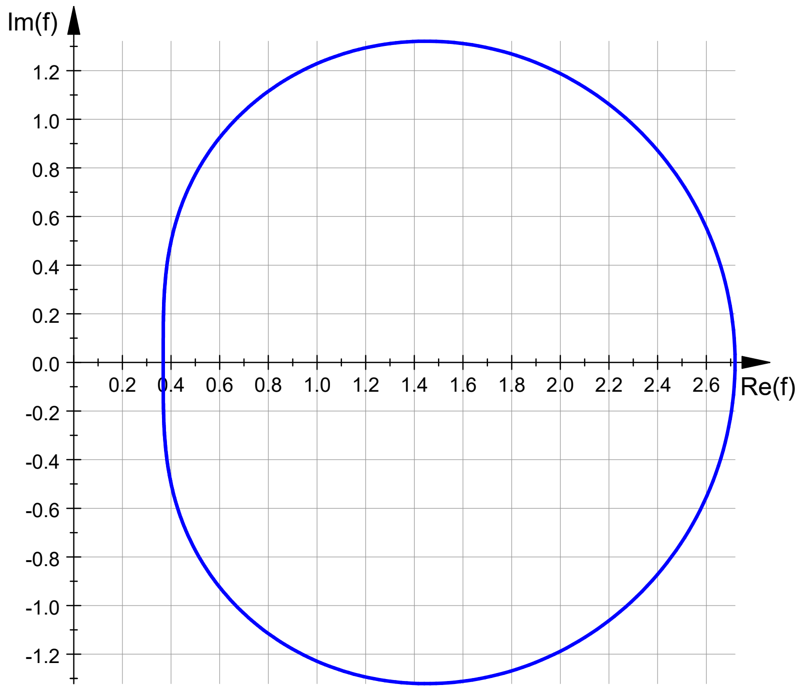


Figure 14: The function $\exp\left(\exp(-x\pi\mathrm{i})\right)$ in the complex plane.

7 In 3D

To transform $f(x)$ into 3D we can define it that way:

$$f_{3D}(x, y) = \frac{1}{x^x y^y} \tag{48}$$

$f_{3D}(x, y)$ is shown in Fig. 15. There is nothing new about it, it just looks nice.

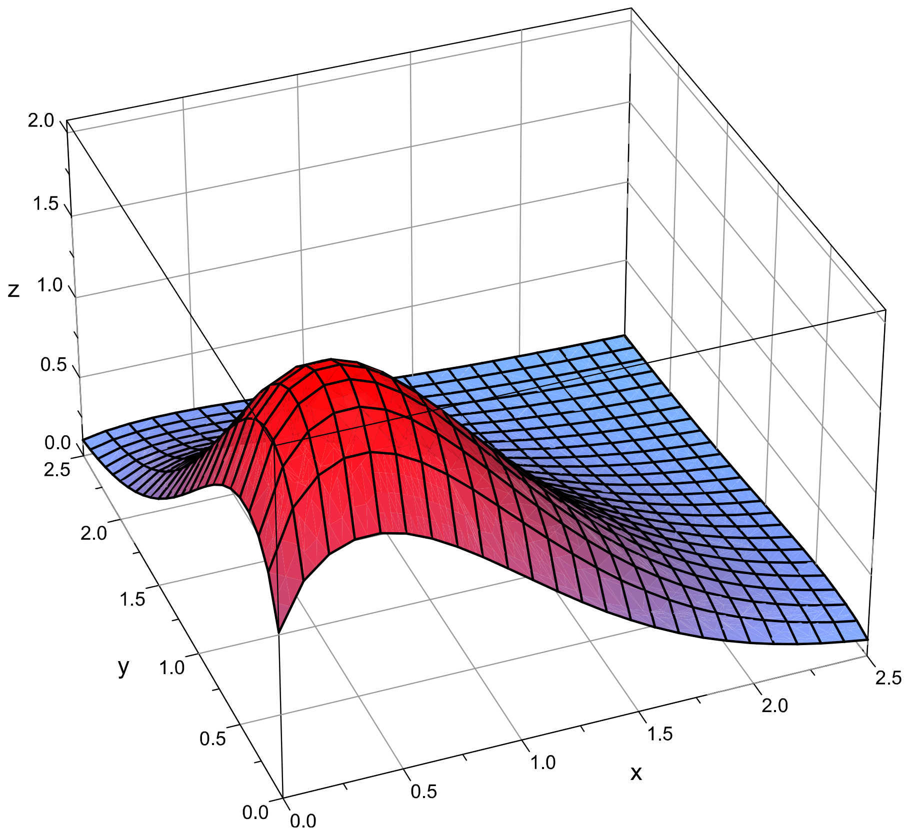


Figure 15: The function $f_{3D}(x, y)$.